DEPARTMENT OF STATISTICS
UNIVERSITY OF WISCONSIN

Madison, Wisconsin 53706

UWIS-DS-80-611

TECHNICAL REPORT NO. 611

May 1980

BOUNDS FOR OPTIMAL CONFIDENCE LIMITS FOR SERIES SYSTEMS.

by

Bernard Harris* Andrew P. Soms**

*University of Wisconsin-Madison

**University of Wisconsin-Milwaukee

Bounds for Optimal Confidence

Limits for Series Systems

Bernard Harris* and Andrew P. Soms**

Abstract

Lindstrom-Madden type approximations to the lower confidence limit on the reliability of a series system are theoretically justified by extending and simplifying the results of Sudakov (1973). Applications are made to Johns (1976) and Winterbottom (1974). Numerical examples are presented.

Key words: Lindstrom-Madden approximation; Optimal confidence bounds; Reliability; Series system.

*University of Wisconsin-Madison.

**University of Wisconsin-Milwaukee. Research supported by the Office of Naval Research under Contract No. N00014-79-C-0321 and the United States Army under Contract No. DAAG29-75-C-0024.

1. Introduction and Summary

A problem of fundamental interest to practitioners in reliability is the statistical estimation of the reliability of a system using experimental data collected on subsystems. In this paper, the subsystem data available consists of a sequence of Bernoulli trials in which a "one" is recorded if the subsystem functions and a zero is recorded if the subsystem fails. Thus for each of the k subsystems composing the system, the data provided consists of the pair (n_i, Y_i), $i=1,2,\ldots k$, where Y_i is binomially distributed (n_i, p_i). We assume that $Y_1, Y_2, \ldots, Y_k$ are mutually independent random variables.

The magnitude of interest in this problem is easily evidenced by the extensive literature devoted to it. In this regard, see the survey paper by Harris (1977) and Section 10.4 of the book by Mann, Schafer, and Singpurwalla (1974). In addition, the Defense Advanced Research Projects Agency has recently issued a Handbook for the Calculation of Lower Statistical Confidence Bounds on System Reliability (1980).

Historically, the first significant work on this problem was produced by Buehler (1957). However, Buehler's method as described in that paper is difficult to implement computationally when $k>2$.

We proceed by describing Buehler's method in Section 2. In Section 3 we specialize to series systems, that is, a system which fails whenever at least one subsystem fails. Sudakov's (1974) results are extended in Section 4 and employed to exhibit some optimality properties of the Lindstrom-Madden method (see Lloyd and Lipow (1962)) for constructing lower confidence bounds for the reliability of series systems of stochastically independent subsystems. Some numerical examples are given in Section 5 and the results needed for this generalization of Sudakov's Theorem are provided in the Appendix to this paper.

2. Buehler's Method for Lower Confidence Bounds

A system composed of k independent subsystems is said to be a coherent system (with respect to the specified decomposition into subsystems), if the system fails when all subsystems fail and the system functions when all subsystems function; and replacing a defective subsystem by a functioning subsystem can not cause a functioning system to fail. Coherent systems are described in Birnbaum, Esary and Saunders (1961) and Barlow and Proschan (1975).

To any system one can associate a function, $h(\tilde{p}) = h(p_1,p_2,\ldots,p_k)$, $0\le p_i\le 1$, $i=1,2,\ldots,k$, where $h(\tilde{p})$ is the reliability of the system when p_i is the probability that the $i^{\underline{th}}$ subsystem functions. It is well-known that if the system is coherent,

$$0 \le h(\tilde{p}) \le 1 ,$$

$$h(0,\ldots,0) = 0,\ h(1,\ldots,1) = 1 ,$$

and $h(p_1,\ldots,p_k)$ is non-decreasing in each variable.

For coherent systems, Buehler's method may be described as follows: The observed outcome $(y_1,\ldots,y_k)$ can assume any of $N = \prod_{i=1}^{k} (n_i+1)$ values, since $y_i = 0,1,\ldots,n_i$. For convenience, we denote n_i-y_i by x_i, $i=1,2,\ldots,k$.

A partition $(A_1,A_2,\ldots,A_s)$, $s>1$, of the N possible outcomes is said to be a monotonic partition, that is, $A_1<A_2<\ldots<A_s$ if $(0,0,\ldots,0) \in A_1$, $(n_1,n_2,\ldots,n_k) \in A_s$ and if $\tilde{x}_1 = (x_{11},\ldots,x_{1k})$, $\tilde{x}_2 = (x_{21},\ldots,x_{2k})$ with $x_{1i} \leq x_{2i}$, $i=1,2,\ldots,k$, then $\tilde{x}_1 \in A_i$ implies $\tilde{x}_2 \in A_j$, $j \geq i$.

Let

$$f(\tilde{x};\tilde{p}) = p_{\tilde{p}}(\tilde{X}=\tilde{x}) = \prod_{i=1}^{k}\binom{n_i}{x_i}p_i^{n_i-x_i}q_i^{x_i} = \prod_{i=1}^{k}\binom{n_i}{y_i}p_i^{y_i}q_i^{n_i-y_i} \tag{2.1}$$

and for $1 \leq n \leq s-1$, let

$$a_n = \inf\left\{h(p)\,\middle|\,\sum_{x_i \in A_i,\, i\leq n} f(\tilde{x}_i;\tilde{p}) = \alpha\right\} \tag{2.2}$$

and $a_s = 0$.

Each such partition may be identified with a function defined on the set of sample outcomes by defining the ordering function $g(\tilde{x})$, where

$$g(\tilde{x}) = n \quad \text{if} \quad \tilde{x} \in A_n\,, \quad 1\leq n\leq s\ ; \tag{2.3}$$

obviously $g(\tilde{x})$ inherits the monotonicity properties of the partition.

Subsequently it will be convenient to use ordering functions $g(\tilde{x})$ such that the range of $g(\tilde{x})$ will be a finite set of real numbers, $r_1<r_2<\ldots<r_s$. With no loss of generality, we can identify the sets A_i by defining $A_i = \left\{\tilde{x}\,|\,g(\tilde{x}) = r_i\right\}$, $i=1,2,\ldots,s$. We can now establish the following theorems.

<u>Theorem 2.1</u>. Let $\tilde{X}$ be distributed by (2.1). Then $a_{g(\tilde{X})}$ is a $(1-\alpha)$ lower confidence bound for $h(\tilde{p})$. If $b_{g(\tilde{X})}$ is also a $(1-\alpha)$ lower confidence bound for $h(\tilde{p})$, then $b_i\leq a_i$, $1\leq i\leq s$.

<u>Proof</u>: Fix $\tilde{p}$ and let $n(\tilde{p})$ be the smallest integer such that

$$P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{n(\tilde{p})} A_i\right\} \geq \alpha , \tag{2.4}$$

and

$$P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=n(\tilde{p})}^{s} A_i\right\} \geq 1-\alpha . \tag{2.5}$$

Let

$$D_n = \left\{\tilde{p} \middle| P_{\tilde{p}}\left\{\tilde{X} \in \bigcup_{i=1}^{n} A_i\right\} \geq \alpha\right\} . \tag{2.6}$$

Then $D_{g(\tilde{X})}$ is a 1-α confidence set for $\tilde{p}$, since

$$P_{\tilde{p}}\left\{\tilde{p} \in D_{g(\tilde{X})}\right\} = P_{\tilde{p}}\left\{g(\tilde{X}) \geq n(\tilde{p})\right\} \geq 1-\alpha . \tag{2.7}$$

This establishes the first part of the conclusion. Further, since $h(\tilde{p})$ is continuous and $0 \leq p_i \leq 1$, the infimum in (2.2) is attained.

Now assume that i_1 is the smallest index such that $b_{i_1} > a_{i_1}$, $1 \leq i_1 \leq s-1$. Then, for some $\tilde{p}_o$, $\tilde{p}_1$,

$$b_{i_1} > \inf\left\{h(\tilde{p}) \middle| \sum_{x_i \in A_i, i \leq i_1} f(\tilde{x};\tilde{p}) = \alpha\right\} = h(\tilde{p}_o) ,$$

and

$$\sum_{x_i \in A_i, i \leq i_1} f(\tilde{x};\tilde{p}_1) > \alpha,\ h(\tilde{p}_1) < b_{i_1} .$$

Therefore

$$P_{\tilde{p}_1}\left\{h(\tilde{p}_1) < b_{g(\tilde{X})}\right\} \geq \sum_{\tilde{x}_i \in A_i, i \leq i_1} f(\tilde{x};\tilde{p}_1) > \alpha,$$

a contradiction.

Remark. Let $d_n = \sup\left\{1-h(\tilde{p}) \middle| \sum_{x_i \in A_i, i \leq n} f(\tilde{x}_i;\tilde{p}) = \alpha\right\}$. Then d_n is a $(1-\alpha)$ upper confidence bound for $1-h(\tilde{p})$, the unreliability.

Let $A = \left\{\tilde{x} \in E_k,\ 0 \leq x_i < a_i,\ i=1,2,\dots,k\right\}$ and let $g(\tilde{x})$ be continuous on $\overline{A}$ (the closure of A) and strictly increasing in each variable for $\tilde{x}\epsilon A$. $g(\tilde{x})$ is to be regarded as an ordering function as described immediately preceding Theorem 2.1. We require the following additional property of $g(\tilde{x})$.

Fix $\tilde{x}_o \epsilon A$. Let $g(\tilde{x}_o) < g(a_1,0,\ldots,0) = g_1$. Then $g(y_1,0,\ldots,0) = g(\tilde{x}_o)$ has a unique solution in y_1. Proceeding recursively, let $i_1 \le y_1$ and define $y_2 = y_2(i_1)$ as the solution of $g(\tilde{x}_o) = g(i_1,y_2\ 0,\ldots,0)$. For each $1\le j\le k$ and $i_{j-1} \le y_{j-1}$, $i_{j-2} \le y_{j-2},\ldots,\ i_1 \le y_1$, let $y_j = y_j(i_1,i_2,\ldots,i_{j-1})$ be the solution of

$$g(\tilde{x}_o) = g(i_1,i_2,\ldots,i_{j-1},y_j,0,\ldots,0) \ . \tag{2.8}$$

We require that the equations indicated in (2.8) have unique solutions for each y_j.

Then define

$$F(\tilde{x}_o;\tilde{p}) = \sum_{i_1=0}^{[y_1]} \sum_{i_2=0}^{[y_2]} \cdots \sum_{i_k=0}^{[y_k]} f(\tilde{i};\tilde{p}) \ , \tag{2.9}$$

where, for $j>1$, $y_j = y_j(i_1,i_2,\ldots,i_{j-1})$. Let

$$f^*(\tilde{x}_o;a) = \sup_{h(\tilde{p})=a} F(\tilde{x}_o;\tilde{p}) \ , \quad 0<a<1 \ . \tag{2.10}$$

Then we have

<u>Theorem 2.2</u>. If $\tilde{x}_o$ satisfies $\inf\limits_{0<a<1} f^*(x_o;a) = 0$, $\sup\limits_{0<a<1} f^*(x_o;a) = 1$ and $f^*(x_o;a)$ is a strictly increasing function of a, and if $\tilde{x}_o \ \epsilon \ A_n$ where $g(\tilde{x})$ determines $(A_1,A_2,\ldots,A_s)$, and if

$$b = \inf\left\{h(\tilde{p}) \,\middle|\, \sum_{x\epsilon A_i,\, i\le n} f(\tilde{x}_i,\tilde{p}) = \alpha\right\} \ , \tag{2.11}$$

then we have

$$f^*(\tilde{x}_o;b) = \alpha \ .$$

Proof: Since the infimum in (2.11) is attained, there is a $\tilde{p}_o$ such that $b = h(\tilde{p}_o)$ and $F(\tilde{x}_o;\tilde{p}_o) = \alpha$. Then $f^*(\tilde{x}_o,b) \geq \alpha$. If $f^*(\tilde{x}_o,b) > \alpha$, there exists $\tilde{p}_a$, with $a = h(\tilde{p}_a)$, $a<b$ and $f^*(\tilde{x}_o;a) = \alpha$ contradicting (2.11).

Obviously, the above discussion can easily be modified to obtain upper confidence bounds on the unreliability $1-h(\tilde{p})$ by replacing inf by sup in (2.11) and requiring that $f^*(\tilde{x}_o;a)$ be a strictly decreasing function of a, $0<a<1$.

3. Applications to Series Systems

For a series system $h(\tilde{p}) = \prod_{i=1}^{k} p_i$. Further, throughout this section we assume that $g(\tilde{x})$ satisfies the conditions necessary to insure that the solutions for $y_1,\ldots,y_k$ indicated in (2.8) are unique. Then we have the following theorem.

Theorem 3.1. If $h(\tilde{p}) = \prod_{i=1}^{k} p_i$, then $\inf_{0<a<1} f^*(\tilde{x}_o;a) = 0$, $\sup_{0<a<1} f^*(\tilde{x}_o,a) = 1$ and $f^*(\tilde{x}_o;a)$ is strictly increasing in a, whenever $\tilde{x}_o = (x_{o1},\ldots,x_{ok})$ satisfies $x_{oj} < n_j$, $j=1,2,\ldots,k$.

Proof. Since $h(\tilde{p}) = 1$ if and only if $p_i = 1$, $i=1,2,\ldots,k$, it follows from (2.1) that

$$\lim_{a\to 1} \sup_{h(\tilde{p})=a} F(\tilde{x}_o;\tilde{p}) = 1 .$$

Similarly, $h(\tilde{p}) = 0$ if and only if at least one $p_i = 0$, $i=1,2,\ldots,k$. Since $F(\tilde{x}_o;\tilde{p}) \leq P_{\tilde{p}}\left\{X_i<n_i\right\} = 1-P_{\tilde{p}}\left\{X_i=n_i\right\} = 1-q_i^{n_i}$, we have

$$\lim_{a\to 0} \sup_{h(\tilde{p})=a} F(\tilde{x}_o;\tilde{p}) = 0 .$$

To show that $f^*(\tilde{x}_o;a)$ is strictly increasing in a, consider

$0<a<b<1$ and let $\tilde{p}_a = (p_{a1},\ldots,p_{ak})$ satisfy $f^*(\tilde{x}_o;a) = F(\tilde{x}_o;\tilde{p}_a)$. Similarly, let $\tilde{p}_b$ satisfy $f^*(\tilde{x}_o;b) = F(\tilde{x}_o;\tilde{p}_b)$. Let $I = \left\{i_1,i_2,\ldots,i_r\right\}$ be any non-empty set of indices such that $p_{ai_j}(\frac{b}{a})^{1/r} < 1$ and let I^c be the remaining indices. Then

$$\left(\prod_{j\in I} p_{ai_j}\left(\frac{b}{a}\right)^{1/r}\right)\prod_{j\in I^c} p_{ai_j} = b \; . \tag{3.1}$$

From the monotone likelihood ratio property of the binomial distribution,

$$F(\tilde{x}_o;\tilde{p}_a) < F(\tilde{x}_o;\tilde{p}^*) \; ,$$

where the components of p^* are given by (3.1). Then

$$F(\tilde{x}_o;\tilde{p}^*) \leq \sup_{h(\tilde{p})=b} F(\tilde{x}_o;\tilde{p}) = F(\tilde{x}_o;\tilde{p}_b) = f^*(x_o;b) \; .$$

4. Sudakov's Method

Let

$$I_p(r,s) = \frac{1}{B(r,s)}\int_0^p t^{r-1}(1-t)^{s-1}dt \; .$$

Then if y is an integer, $y<n$, we have

$$\sum_{i=0}^{y} \binom{n}{i} p^{n-i}q^{i} = I_p(n-y,y+1) \; .$$

For $0\leq y<n$, real, define $u(n,y,\alpha)$ by $\alpha = I_{u(n,y,\alpha)}(n-y,y+1)$. Thus, for integer values of y, $u(n,y,\alpha)$ is a $100(1-\alpha)$ percent lower confidence limit for p. Sudakov (1973) showed that for $n_1\leq n_2\leq\cdots\leq n_k$ and $g(\tilde{x}) = \prod_{i=1}^{k}(n_i-x_i)$,

$$u(n_1,y_1,\alpha) \leq b \leq u(n_1,[y_1],\alpha) \; .$$

where $$y_1 = n_1 q_0,\ q_0 = 1 - \prod_{i=1}^{k} ((n_i - x_{0i})/n_i) .$$

$u(n_1, y_1, \alpha)$ is called the Lindstrom-Madden method for determining lower confidence limits for the reliability of series systems (see Lloyd and Lipow (1962)).

Lipow and Riley (1959) used a different ordering function; nevertheless they noted that for "small" n_i, their tabulated values provided good agreement with the results using the Lindstrom-Madden method. For large values of n_i, the tabulated values that they provided are based on the Lindstrom-Madden method. Here we provide a further justification for the Lindstrom-Madden method by establishing that it provides conservative lower confidence limits (i.e. is a lower bound to b defined in (2.9)) using the ordering function $g(\tilde{x})$ employed by Sudakov and we also obtain an upper bound for b, thus determining the possible error of the Lindstrom-Madden method.

Sudakov's proof is unnecessarily complicated and contains some incorrect assertions, which nevertheless do not affect the validity of the conclusion. In the Appendix we provide a simpler proof of some auxiliary results needed for the generalization of Sudakov's theorem given below.

Theorem 4.1. Let $g(\tilde{x})$ satisfy the hypothesis of Theorem 3.1. Then,

$$b \leq \min_{1 \leq i \leq k} u(n_i, [y_i^*], \alpha) . \tag{4.1}$$

where b is given by (2.11) and $y_i^* = y_i(j_1, j_2, \ldots, j_{i-1})$ is evaluated at $j_\ell = 0$, $\ell = 1, 2, \ldots, i-1$. Note that $y_1 = y_1^*$. If we also have

$$\frac{y_j - i_j}{n_j - i_j} \geq \frac{y_{j+1}}{n_{j+1}} , \quad j=1,2,\ldots,k-1 , \tag{4.2}$$

then

$$u(n_1, y_1, \alpha) \leq b . \tag{4.3}$$

Proof: (4.1) is immediate from (2.11) upon setting $p_j = 1$, $j \neq i$ and solving $F(\tilde{x}_o; 1,\ldots,1,p_i,1,\ldots,1) = \alpha$. Recall that $n_1 \leq n_2 \leq \ldots \leq n_k$ and

$$F(\tilde{x}_o;\tilde{p}) = \sum_{i_1=0}^{[y_1]} b(n_1 - i_1; p_1, n_1)$$

$$\ldots \sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_k - [y_k], [y_k]+1) . \tag{4.4}$$

Now, apply Lemmas A1, A2, and A3 to the innermost sum in (4.4), to get

$$\sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_k - [y_k], [y_k]+1) \leq$$

$$\sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_k - y_k, y_k+1) \leq$$

$$\sum_{i_{k-1}=0}^{[y_{k-1}]} b(n_{k-1} - i_{k-1}; p_{k-1}, n_{k-1}) I_{p_k}(n_{k-1} - y_{k-1}, y_{k-1} - i_{k-1}+1) \leq$$

$$I_{p_{k-1}p_k}(n_{k-1} - y_{k-1}, y_{k-1}+1) .$$

Repeated applications of the above establish that

$$F(\tilde{x}_o;\tilde{p}) \leq I_{\prod_{i=1}^{k} p_i}(n_1 - y_1, y_1+1) . \tag{4.5}$$

(4.3) follows immediately from (4.5), completing the proof.

Remarks. If (4.3) holds and y_1 is an integer, then $b = f(n_1, y_1, \alpha)$.

It has often been suggested (Lloyd and Lipow (1962), Winterbottom (1974), Bolshev and Loginov(1966), Miraiy and Solov'yev (1964)) that the confidence level should depend only on n_1 the smallest sample size. We now provide a numerical illustration to show that the bound in (4.1) may be improved by taking all the n_i's into consideration.

Let $k=3$, $\alpha=.1$, $\tilde{n} = (10,12,30)$, $\tilde{x}_o = (0,3,0)$. Then for $g(\tilde{x}) = \prod_{i=1}^{3} (n_i - x_i)$, $f(n_1,[y_1],\alpha) = .541$, $f(n_2,[y_2],\alpha) = .525$, $f(n_3,[y_3],\alpha) = .639$. The use of (4.3) establishes $.500 \le b \le .525$.

Note that if $x_{oi} = n_i$, for some i, $1 \le i \le k$, then $g(\tilde{x}) = 0$ and $b=0$. It seems reasonable to use $b=0$ as the lower confidence limit whenever $x_{oi} = n_i$ for any monotone ordering function satisfying the conditions of Section 2.

We now show that if $g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i)$, then (4.2) is satisfied and Theorem 4.1 applies. This result will extend a result due to Winterbottom (1974), who established this fact for particular special cases. In addition, we will also show that (4.2) holds for a number of other ordering functions used in the literature.

Theorem 4.2. Let $g(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i + \alpha_i)$, where $\alpha_i \ge 0$ and $n_{i+1}\alpha_i \ge \alpha_{i+1} n_i$, $i=1,2,\ldots,k-1$. Then (4.2) is satisfied.

Proof. If

$$(n_i - y_i + \alpha_i) \prod_{j=i+1}^{k} (n_j + \alpha_j) = c$$

and

$$(n_i - k_i + \alpha_i)(n_{i+1} - y_{i+1} + \alpha_{i+1}) \prod_{j=i+2}^{k} (n_j + \alpha_j) = c \ ,$$

then we have

$$(n_i-y_i+\alpha_i)(n_{i+1}+\alpha_{i+1}) = (n_i-k_i+\alpha_i)(n_{i+1}-y_{i+1}+\alpha_{i+1}) \ .$$

establishing

$$\frac{y_i-k_i}{n_i-k_i} = \frac{y_{i+1}}{n_{i+1}} \frac{n_{i+1}(n_i+\alpha_i-k_i)}{(n_{i+1}+\alpha_{i+1})(n_i-k_i)} \ .$$

Thus (4.2) holds if

$$\frac{n_{i+1}(n_i+\alpha_i-k_i)}{(n_{i+1}+\alpha_{i+1})(n_i-k_i)} \geq 1 \ ;$$

this last inequality will be true whenever $n_{i+1}\alpha_i \geq \alpha_{i+1}n_i$. In particular, this is valid when $\alpha_i=0$, $i = 1,\ldots,k$ which is Sudakov's ordering function.

Theorem 4.3. If $g(\tilde{x}) = 1 - \sum_{i=1}^{k} x_i/n_i$, then (4.2) is satisfied.

Proof. If $1 - y_i/n_i = c = 1 - \frac{k_i}{n_i} - \frac{y_{i+1}}{n_{i+1}}$, then

$$\frac{y_i-k_i}{n_i} = \frac{y_{i+1}}{n_{i+1}}$$

or

$$\frac{y_i-k_i}{n_i-k_i} \geq \frac{y_{i+1}}{n_{i+1}} \ .$$

This type of ordering function has been employed by Pavlov (1973), for example.

Theorem 4.4. Let $g(\tilde{x}) = \sum_{i=1}^{k} a_i x_i + z_\alpha (a_i^2 x_i)^{\frac{1}{2}}$, where z_α satisfies $1-\phi(z_\alpha) = \alpha$ and $\phi(x)$ is the standard normal distribution function, $a_1 \geq a_2 \geq \cdots \geq a_k$, and $a_i = (n_i \sum_{i=1}^{k} 1/n_i)^{-1}$. Then $g(\tilde{x})$ satisfies (4.2) if and only if

$$(a_j - a_{j+1}) y_j \geq (a_j - a_{j+1}) z_\alpha^2 + a_j k_j - \alpha_j k_j (z_\alpha^2 a_j + 2c - a_j(y_j + k_j)) . \qquad (4.6)$$

Proof: If $g(\tilde{x}_0) = c + \sum_{i=1}^{j-1} a_i k_i$, then defining $\sum_{i=1}^{j-1} a_i^2 k_i = c_1$,

$$a_j y_j + z_\alpha (c_1 + a_j^2 y_j)^{\frac{1}{2}} = c \qquad (4.7)$$

and

$$a_j k_j + a_{j+1} y_{j+1} + z_\alpha (c_1 + a_j^2 k_j + a_{j+1}^2 y_{j+1})^{\frac{1}{2}} = c . \qquad (4.8)$$

Equating the left hand sides of (4.7) and (4.8), we obtain (4.6). If k=2, (4.6) holds for all cases of interest.

If (4.6) holds, then setting

$$1-\alpha = (\Gamma(x))^{-1} \int_0^{f(x,1-\alpha)} t^{x-1} e^{-t} dt ,$$

a straightforward limiting argument shows that

$$\max_i a_i f([y_i]+1, 1-\alpha) \leq b \leq a_1 f(y_1+1, 1-\alpha) . \qquad (4.9)$$

This ordering function has been used by Johns (1976) and b in (4.7) is the value tabulated by Johns for k=2. The validity of the lower bound does not depend on (4.6). In Table 1 below, the lower and upper bounds given in (4.9) are tabulated along with the values given by Johns for $\alpha=.1$. These refer to upper confidence limits for the Poisson parameter combinations $a_1\lambda_1 + a_2\lambda_2$.

Note in particular that three of the values tabulated by Johns (indicated by asterisks) violate (4.9). Specifically consider 5.24, in which case $[y_1] = 5$, since $g^*(2,5) = 4.78$, $g^*(5,0) = 4.72$ and $g^*(6,0) = 5.48$. Using the Poisson approximation we obtain the value 9.275 for the upper confidence limit to λ for $\alpha=.1$ and thus $a_1\lambda_1 + a_2\lambda_2 = 5.56$. Consequently the sup must exceed 5.56. An alternative approach to the one suggested by Johns for $k \geq 3$ is to simply use $a_1 f(y_1+1, 1-\alpha)$ for b.

Table 1

Comparison of Upper and Lower Bounds
With Values Tabulated by Johns for $\alpha=.1$

a_1	x_1	x_2	Lower Bound	Upper Bound	Johns' Tabled Value
.9	7	2	4.79	5.50	5.17
.9	3	0	2.07	2.27	2.16
.75	6	3	6.00	6.65	6.23
.75	12	3	7.90	8.29	7.91
.67	3	3	5.36	5.61	5.33*
.67	15	2	8.71	9.24	8.81
.60	5	2	5.56	5.62	5.24*
.60	7	6	9.24	9.53	9.18*

5. Numerical Examples and Concluding Remarks

Examples 1 and 2 illustrate the method we have described in this paper.

<u>Example 1</u>: Let $H(\tilde{x}) = \prod_{i=1}^{k} (n_i - x_i)$, $\alpha = .05$, $k = 5$, $\tilde{n} = (20,30,40,25,60)$, $\tilde{x} = (2,6,10,8,15)$. Then the 95% upper confidence limit for the failure probability is contained in (.86,.88).

<u>Example 2</u>: Let $H(\tilde{x}) = \prod(n_i - x_i)$, $\alpha = .05$, $k = 2$, $\tilde{n} = (10,10)$, $\tilde{x} = (3,2)$. Then the 95% upper confidence limit for the failure probability is contained in (.70, .73). The value given in Lipow and Riley (1959) is .70.

<u>Remarks</u>. In this paper we have showed that the Lindstrom-Madden technique is conservative for ordering functions satisfying (4.2).

Further, if y_1 is an integer, then the Lindstrom-Madden method is exact. We have also relaxed the conditions needed in Winterbottom (1974) and provided an alternative to the method of Johns (1976).

Appendix

The auxiliary results employed in the proof of Theorem 4.1 are provided here.

Lemma A1: $I_y(n-x,x+1)$, $0\leq y\leq 1$, is a decreasing function of n and an increasing function of x. $I_y(np,nq+1)$, $p+q=1$, $0<p<1$, is an increasing function of q.

Proof: The proof is immediate from the observation that the beta distribution with parameters α and β has monotone likelihood ratio in α and $-\beta$ and that if a probability distribution has monotone likelihood ratio in θ, $F_\theta(x)$ is a decreasing function of θ (Lehmann (1959), p. 68 and p. 74).

Lemma A2: If $\frac{y_i-k_i}{n_i-k_i}\geq\frac{y_{i+1}}{n_{i+1}}$ and $n_i\leq n_{i+1}$, then

$$I_y(n_i-y_i,y_i-k_i+1)\geq I_y(n_{i+1}-y_{i+1},y_{i+1}+1)\ . \tag{A.1}$$

Proof: Rewriting the left and right hand sides of (A.1) as

$$I_y\left[(n_i-k_i)(1-\frac{y_i-k_i}{n_i-k_i}),\ (n_i-k_i)(\frac{y_i-k_i}{n_i-k_i})+1\right]\geq$$

$$I_y\left[n_{i+1}(1-\frac{y_{i+1}}{n_{i+1}}),\ n_{i+1}(\frac{y_{i+1}}{n_{i+1}})+1\right]\ . \tag{A.2}$$

Lemma A1 applies and the conclusion follows.

Lemma A3: Let $y_1y_2=y$, $0\leq y_i\leq 1$, $i=1,2$. Then

$$I_{y_1y_2}(n-x,x+1)\geq\sum_{k=0}^{[x]}b(n-k;y_1,n_1)I_{y_2}(n-x,x-k+1)\ . \tag{A.3}$$

Proof:

$$\sum_{k=0}^{[x]} \binom{n}{k} y_1^{\,n-k}(1-y_1)^k \frac{\Gamma(n-k+1)}{\Gamma(n-x)\Gamma(x-k+1)} \int_0^{y_2} t^{n-x-1}(1-t)^{x-k}dt$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-x)} \sum_{k=0}^{[x]} \frac{(1-y_1)^k y_1^{\,n-k}}{k!\Gamma(x-k+1)} \int_0^{y_1y_2} \left(\frac{t}{y_1}\right)^{n-x-1} \left(\frac{y_1-t}{y_1}\right)^{x-k} \frac{dt}{y_1}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n-x)} \int_0^{y_1y_2} \sum_{k=0}^{[x]} \frac{(1-y_1)^k t^{n-x-1}(y_1-t)^{x-k}}{k!\Gamma(x-k+1)}\, dt\ .$$

Thus (A.3) will hold whenever

$$\frac{\Gamma(n+1)}{\Gamma(n-x)\Gamma(x+1)} \int_0^{y_1y_2} t^{n-x-1}(1-t)^x dt \geq$$

$$\frac{\Gamma(n+1)}{\Gamma(n-x)} \int_0^{y_1y_2} \sum_{k=0}^{[x]} \frac{(1-y_1)^k t^{n-x-1}(y_1-t)^{x-k}}{k!\ \Gamma(x-k+1)}\, dt$$

or

$$\frac{\Gamma(n+1)}{\Gamma(n-x)\Gamma(x+1)} \int_0^{y_1y_2} t^{n-x-1}(1-t)^x \cdot$$

$$\left[1-\sum_{k=0}^{[x]} \frac{\Gamma(x+1)}{k!\Gamma(x-k+1)} \left(\frac{1-y_1}{1-t}\right)^k \left(\frac{y_1-t}{1-t}\right)^{x-k}\right] dt \geq 0\ . \qquad \text{(A.4)}$$

Writing $\frac{y_1-t}{1-t} = \left(1 - \frac{1-y_1}{1-t}\right)$ and noting that $0 \leq t \leq y_1y_2 \leq 1$, $t<y_1$ and $(1-t)>1-y_1$, we observe that (A.4) holds and the lemma is proved.

References

Barlow, Richard E. and Proschan, Frank (1975), Statistical Theory of Reliability and Life Testing, Probability Models, New York: Holt, Rinehart and Winston.

Birnbaum, Z.W., Esary, James D., and Saunders, Sam C. (1961), "Multicomponent Systems and Their Reliability", Technometrics, 3, 55-77.

Bol'shev, L.N. and Loginov, E.A. (1966), "Interval Estimates in the Presence of Noise", Theory of Probability and Its Applications, 11, 82-94.

Buehler, Robert J. (1957), "Confidence Limits for the Product of Two Binomial Parameters", Journal of the American Statistical Association, 52, 482-93.

Defense Advanced Research Projects Agency (1980), Handbook for the Calculation of Lower Statistical Confidence Bounds on System Reliability.

Harris, Bernard (1977), "A Survey of Statistical Methods in Systems Reliability Using Bernoulli Sampling of Components", in Theory and Applications of Reliability: With Emphasis on Bayesian and Nonparametric Methods, eds. Chris P. Tsokos and I.N. Shimi, New York: Academic Press.

Johns, M.V., Jr. (1976), "Confidence Bounds for Highly Reliable Systems", Unpublished technical report, Department of Statistics, Stanford University.

Lehmann, E.L. (1959), Testing Statistical Hypotheses, New York: John Wiley and Sons.

Lipow, M. and Riley, J. (1959), "Tables of Upper Confidence Bounds on Failure Probability of 1, 2, and 3 Component Serial Systems", Vols. I and II, Space Technology Laboratories.

Lloyd, D.K. and Lipow, M. (1962), Reliability: Management, Methods, and Mathematics, Englewood Cliffs: Prentice Hall.

Mann, Nancy R., Schafer, Ray E., and Singpurwalla, Nozer D. (1974), Methods for Statistical Analysis of Reliability and Life Data, New York: John Wiley and Sons.

Mirniy, R.A. and Solov'yev, A.D. (1964), "Estimation of the Reliability of a System from the Results of Tests of its Components", Kibernetiky na Sluzhby Kommunizmy, 2, Energiya, Moscow.

Pavlov, I.V. (1973), "A Confidence Estimate of System Reliability from Component Testing Results", Izvestiya Akad. Nauk. Tech. Kibernetiky, 3, 52-61.

Sudakov, R.S. (1974), "On the Question of Interval Estimation of the Index of Reliability of a Sequential System", Engineering Cybernetics, 12, 55-63.

Winterbottom, Alan (1974), "Lower Limits for Series System Reliability from Binomial Data", Journal of the American Statistical Association, 69, 782-8.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE — READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: TR #611
2. GOVT ACCESSION NO.: AD-A085741
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): Bounds for Optimal Confidence Limits for Series Systems
5. TYPE OF REPORT & PERIOD COVERED: Scientific-Interim
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): Bernard Harris and Andrew P. Soms
8. CONTRACT OR GRANT NUMBER(s): Contract No. N00014-79-C-0321
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Statistics, University of Wisconsin, Madison, Wisconsin 53706
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS:
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, 800 N. Quincy Street, Arlington, VA 22217
12. REPORT DATE: May 21, 1980
13. NUMBER OF PAGES: 19 pages
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS (of this report): Unclassified
15a. DECLASSIFICATION DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Distribution of this document is unlimited
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
   - Lindstrom-Madden approximation
   - Optimal confidence bounds
   - Reliability
   - Series system
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

Lindstrom-Madden type approximations to the lower confidence limit on the reliability of a series system are theoretically justified by extending and simplifying the results of Sudakov (1973). Applications are made to Johns (1976) and Winterbottom (1974). Numerical examples are presented.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)